

# Is the 126 GeV Higgs Boson Mass Calculable in Gauge-Higgs Unification?

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## Abstract

We address a question whether the recently observed Higgs mass  $M_H = 126$  GeV, of the order of the weak scale  $M_W$ , is calculable as a finite value in the scenario of gauge-Higgs unification. In the scenario formulated on a flat 5-dimensional space-time, the Higgs mass is calculable, being protected under the quantum correction by gauge invariance, though the predicted Higgs mass is generally too small compared with  $M_W$ . In the 6-dimensional SU(3) model, however, a suitable orbifolding is known to lead to a mass of the order of  $M_W$ :  $M_H = 2M_W$  at the tree level, which has some similarity to the corresponding prediction by the MSSM,  $M_H \leq (\cos \beta)M_Z$ .

We demonstrate first by a general argument and secondly by explicit calculations that, even though the quantum correction to the quartic self-coupling of the Higgs field is UV-divergent, its deviation from that of  $g^2$  is calculable, and therefore two observables,  $M_H^2$  and  $\Delta \equiv (\frac{M_H}{2M_W})^2 - 1$ , are both calculable in the gauge-Higgs unification scenario. The implication of the precise value 126 GeV to the compactification scale and the bulk mass of the matter field in our model is also discussed.

# 1 Introduction

The discovery of the Higgs particle was a great success of LHC experiment [1, 2]. We, however, should note that the long-standing problems concerning the property of Higgs and its interactions, such as the hierarchy problem, are still there and we do not have any conclusive argument of the origin of the Higgs itself. Many of the theories of physics beyond the standard model (BSM) have been proposed in order to solve the hierarchy problem. At this stage, we do not know whether the discovered scalar particle is really what the standard model predicts or a particle some theory of BSM has in its low energy effective theory.

On the other hand, it is interesting to note that the observed Higgs mass,  $M_H = 126$  GeV, seems to give us some hints on the issues discussed above. Namely, the Higgs mass is roughly of the order of the weak scale  $M_W$  and therefore Higgs has turned out to be relatively “light”. Thus, we may say that the theories predicting light Higgs are favored among proposed BSM theories, if they are ever realized in nature, while strongly coupled Higgs sector seems to be ruled out.

The Higgs mass of  $\mathcal{O}(M_W)$  may also suggest that the Higgs mass is basically handled by gauge interaction. For instance, in MSSM the predicted Higgs mass is not far from the weak scale, since the Higgs quartic coupling  $\lambda$  gets contribution only from gauge interaction (the D-term contribution) and  $\lambda \sim \mathcal{O}(g^2)$  at the tree level.

We may ask a fundamental question: is it ever possible to predict the Higgs mass? In fact, in the SM, the Higgs mass acquires a divergent quantum correction and the observed Higgs mass is only realized by an adjustment of the bare Higgs mass: the origin of the hierarchy problem. Thus  $M_H$  is not predictable. On the other hand, in the MSSM, for instance,  $M_H$  is calculable as a finite value (predictable) even under the quantum correction, since the relation  $\lambda \sim \mathcal{O}(g^2)$  holds at the tree level because of the supersymmetry.

It may be quite interesting to ask ourselves whether there exist other possibilities of BSM theories with predictable Higgs mass  $M_H$ . From such a point of view, in this paper we focus on another interesting scenario of BSM, i.e. “gauge-Higgs unification (GHU)”. In the scenario of GHU, the Higgs field is identified with an extra-space component of higher dimensional gauge field. The scenario itself is not new [3, 4, 5], and it has been pointed out some time ago that the hierarchy problem is solved in this scenario thanks to the higher-dimensional gauge symmetry [6].

Although these scenarios, MSSM and GHU, are completely independent, they have some features in common. First, both aim to solve the hierarchy problem relying on some symmetries. Secondly, also in GHU scenario the Higgs mass is basically controlled by gauge interaction, just because the Higgs is nothing but a gauge field to start with in this scenario. Thus,  $M_H$  is calculable in the GHU. In fact, after [6] the finiteness of the Higgs mass has been demonstrated in various types of models of GHU and even at the two loop level [7].

One basic problem of GHU is that the Higgs potential does not exist at the tree level in the simplest case of 5-dimensional (5D) space-time, as the gauge fields in general have no potential term. Thus  $M_H = 0$  at the tree level. Even though the Higgs mass is induced at the quantum level, it is generally too small,  $M_H^2 = \mathcal{O}(\alpha M_W^2)$ , though it may be lifted once the 5D space-time is assumed to

be a curved Randall-Sundrum type background [8]. The situation may change if the number of the extra space is greater than one. For instance in 6D space-time, the Higgs potential gets a contribution already at the tree level from a term  $g^2[A_5, A_6]^2$  in  $F_{MN}F^{MN}$ , where  $F_{MN}$  is a field strength of the higher dimensional gauge field  $A_M$  ( $M = \mu$  ( $\mu = 0, 1, 2, 3$ ), 5, 6) [9]. The term  $g^2[A_5, A_6]^2$  provides a non-vanishing quartic self-coupling of the Higgs field, unless the  $A_5, A_6$  components of the Higgs field are proportional to each another. In fact, in the 6D GHU model with  $T^2/Z_3$  orbifold as its extra space, the quartic coupling  $\lambda$  exists at the tree level, which is given in terms of the gauge coupling  $g$  as

$$\lambda_{tree} = \frac{1}{2}g^2, \quad (1.1)$$

similarly to the case of MSSM. (1.1) in turn implies that

$$M_H = 2M_W, \quad (1.2)$$

once the Higgs field acquires its VEV ( $M_H^2 = 2\lambda v^2$ ,  $M_W = \frac{1}{2}gv$  ( $v$  : the VEV of the Higgs field)). The situation is quite similar to the case of MSSM, where

$$M_H \leq (\cos \beta)M_Z, \quad (1.3)$$

at the tree level.  $\beta$  is defined as the ratio of two Higgs doublet's VEVs:  $\tan \beta \equiv \langle H_u \rangle / \langle H_d \rangle$ . Thus we expect that in GHU the Higgs mass is calculable as a finite value even after the quantum correction, just as in the case of MSSM.

It is quite interesting to note that both two scenarios of BSM which aim to solve the hierarchy problem, MSSM and GHU, predict the Higgs mass of the order of the weak scale  $M_W$ , being consistent with the observation. So, a natural question to ask next is what the observed precise value of the Higgs mass,  $M_H = 126$  GeV, implies for these scenarios. Note that in MSSM the observed Higgs mass is explained by choosing a suitable SUSY-breaking mass scale  $M_{SUSY}$ , though the required  $M_{SUSY}$  is claimed to be a little too high from the view point of the hierarchy problem.

Actually, the quantum correction to the Higgs mass in MSSM is much larger than we naive expect as the quantum correction:  $M_Z + 35$  GeV  $\simeq 126$  GeV, which is comparable to the weak scale itself. Surprisingly, in the GHU if the quantum correction of the same size is realized, the corrected Higgs mass happens to be just what has been observed:  $2M_W - 35$  GeV  $\simeq 126$  GeV! A relative sign difference of the quantum correction is expected from the difference of spin statistics of the particles running inside the loop in the quantum correction, i.e. stop for the case of MSSM and Kaluza-Klein (KK) top quarks for the case of GHU, for instance. Most probably, the relation mentioned above is just a coincidence, but this at least motivates the study of the quantum correction to the Higgs mass in the GHU.

To be more concrete, what we calculate in this paper is the quantum correction to the following two observables which have been now completely fixed by the recent LHC experiments at CERN:

$$M_H^2, \quad (1.4)$$

$$\Delta \equiv \left( \frac{M_H}{2M_W} \right)^2 - 1. \quad (1.5)$$

Both of  $M_H^2$  and  $\Delta$  turn out to vanish at the tree level in our model of GHU, as is seen from (1.2) in the case of  $\Delta$ . Thus we naturally expect that even after the quantum correction these quantities are non-vanishing but finite, i.e. calculable. We confirm this expectation by explicitly calculating the quantum corrections to  $M_H^2$  and  $\Delta$ , as we will see later.

Let us note that in MSSM, though the ratio of the quartic coupling of the Higgs to  $g^2$  is calculable as a function of  $M_{SUSY}$ , the quadratic term of the Higgs, coming from the “ $\mu$ -term” and SUSY breaking mass-squared term, exists already at the tree level and is not calculable, in contrast to the case of GHU.

To see why these two quantities vanish at the tree level, we concentrate on the part in the lagrangian, relevant for the Higgs and  $W$  boson masses through the spontaneous symmetry breaking:

$$-(-\mu^2|h_0|^2 + \lambda|h_0|^4) + \kappa|h_0|^2 W^{+\mu} W_\mu^-, \quad (1.6)$$

where  $h_0$  is the electrically neutral component of the Higgs doublet. By use of the coefficients  $\mu^2, \lambda, \kappa$ , two observables are expressed as

$$M_H^2 = 2\mu^2, \quad (1.7)$$

$$\Delta = \frac{\lambda}{\kappa} - 1. \quad (1.8)$$

The coefficients at the classical level, denoted as  $\mu_{tree}^2$  etc., are known to be

$$\mu_{tree}^2 = 0, \quad \lambda_{tree} = \kappa_{tree} = \frac{1}{2}g^2. \quad (1.9)$$

Hence both  $M_H^2$  and  $\Delta$  vanish at the tree level. Note that at the tree level the spontaneous symmetry breaking does not occur:  $M_H = M_W = 0$ , keeping the relation  $M_H = 2M_W$ . The reason why the relations in (1.9) hold is that the coefficients  $\mu^2, \lambda$  and  $\kappa$  are all provided by a single operator in the lagrangian, i.e. the kinetic term of the higher-dimensional gauge boson

$$-\frac{1}{2}\text{Tr}(F_{MN}F^{MN}) \quad (M, N = \mu, 5, 6). \quad (1.10)$$

This operator yields the Higgs potential via  $g^2[A_M, A_N]^2$  term, but only its quartic term, not a quadratic term, leading to  $\mu_{tree}^2 = 0$ . On the other hand, the relation between the coefficients of  $g^2[A_5, A_6]^2$  and  $g^2[A_\mu, A_5]^2$ ,  $g^2[A_\mu, A_6]^2$  yields  $\lambda_{tree} = \kappa_{tree}$ .

Our main purpose is to calculate the quantum corrections to  $M_H^2$  and  $\Delta$  and demonstrate explicitly that these two observables are in fact calculable. We also compare the predictions of our model with the experimental data on these two observables obtained by the recent LHC experiments and will discuss whether the observed values can be accounted for by a suitable choice of the parameters of our theory, such as the compactification mass scale  $M_c \equiv 1/R$ , corresponding to  $M_{SUSY}$  in MSSM. Here  $R$  is the size of  $T^2$  of the orbifold.

To be strict, however, we should note that such naive expectation may not necessarily be realized in the non-renormalizable theory like higher dimensional gauge theory, since operators with higher mass dimensions induced at the quantum level may also be UV-divergent. To be more precise, e.g.,

$(F_{MN}F^{MN})^2$  having mass dimension  $d = 8$  (from the viewpoint of 4D space-time) may be harmless, but  $(D_L F_{MN})(D^L F^{MN})$  ( $d = 6$ ) may be potentially dangerous having logarithmic divergence in the quantum correction to the coefficient. Fortunately, we readily find that the operator with  $d = 6$  contributes only to 6-point self-couplings of the Kaluza-Klein (KK) zero-modes of 4D Higgs and gauge bosons, which do not affect the effective lagrangian given in (1.6). Though these “irrelevant” operators still may change the form of the Higgs potential, we expect the contributions are relatively suppressed by higher powers of  $M_W^2/M_c^2$ .

## 2 The model

In the scenario of GHU, the gauge group should be inevitably enlarged. As the simplest choice we choose  $SU(3)$  as the gauge group [10, 11]. Thus, we work in the model where as the matter field scalar fields belonging to an  $SU(3)$  triplet,  $\Phi$ , are introduced in 6D space-time with  $T^2/Z_3$  orbifold as the extra space. The torus  $T^2$  is assumed to have the same period  $2\pi R$  in both directions of two cycles.  $Z_3$  is nothing but a rotation with the angle  $\frac{2\pi}{3}$  in the two-dimensional extra space described by the coordinates  $(x_5, x_6)$ . In this paper we aim to demonstrate that our program to predict the Higgs mass as calculable finite value works by taking a toy model. That is the reason why we adopt scalar fields as the matter fields. In order to make the model realistic we are planning to introduce fermionic matter fields in future study, though the mechanism to get calculable  $M_W$ ,  $\Delta$  will not change, as our argumentation is based on general features of GHU, especially the higher dimensional gauge symmetry.

The lagrangian is given as

$$\mathcal{L} = (D_M \Phi)^\dagger (D^M \Phi) - M^2 \Phi^\dagger \Phi - \frac{1}{2} \text{Tr}(F_{MN} F^{MN}) \quad \left( F_{MN} = F_{MN}^a T^a, \text{Tr}(T^a T^b) = \frac{1}{2} \delta^{ab} \right). \quad (2.1)$$

where the covariant derivative for the triplet scalar is given as

$$D_M = \partial_M + ig A_M \quad (A_M = A_M^a T^a), \quad (2.2)$$

and the bulk mass  $M$  is introduced in order to avoid infra-red divergence appearing in the quantum correction to the coefficient  $\lambda$ , as we will see later.

The  $Z_3$ -parity for the triplet scalar is assigned as follows [12]:

$$\Phi(x, \omega z) = \Theta_0 \Phi(x, z) \quad (z = x_5 + ix_6, \omega = e^{i\frac{2\pi}{3}}), \quad (2.3)$$

where

$$\Theta_0 = \text{diag}(1, 1, \omega). \quad (2.4)$$

Thus only upper two components of the triplet have KK-zero-modes, whose mode function is just a constant. Note that the bulk mass term in (2.1) is  $Z_3$  invariant.

The 6D field  $\Phi$  is expanded in terms of mode-functions as follows

$$\Phi(x, z) = \int \frac{d^4 p}{(2\pi)^4} e^{ipx} \frac{1}{12^{\frac{1}{4}} \pi R} \sum_{n,m} \begin{pmatrix} \phi_{n,m}^{(1)}(p) f_{n,m}^{(0)}(z) \\ \phi_{n,m}^{(2)}(p) f_{n,m}^{(0)}(z) \\ \phi_{n,m}^{(3)}(p) f_{n,m}^{(1)}(z) \end{pmatrix}, \quad (2.5)$$

where the KK-mode-functions are given as

$$\begin{aligned} 1 : \quad & f_{n,m}^{(0)}(z) = f_{n,m}(z) + f_{n,m}(\omega z) + f_{n,m}(\bar{\omega} z), \\ \omega : \quad & f_{n,m}^{(1)}(z) = f_{n,m}(z) + \bar{\omega} f_{n,m}(\omega z) + \omega f_{n,m}(\bar{\omega} z), \\ \bar{\omega} : \quad & f_{n,m}^{(2)}(z) = f_{n,m}(z) + \omega f_{n,m}(\omega z) + \bar{\omega} f_{n,m}(\bar{\omega} z), \end{aligned} \quad (2.6)$$

$$f_{mn}(z) = \exp \left( \frac{i}{2R} \left\{ \left( n - \frac{n+2m}{\sqrt{3}} i \right) z + c.c. \right\} \right). \quad (2.7)$$

Note that each mode function  $f_{n,m}^{(0)}(z)$ ,  $f_{n,m}^{(1)}(z)$ ,  $f_{n,m}^{(2)}(z)$  has a definite eigenvalue under the  $Z_3$  transformation, “ $Z_3$ -parity”, 1,  $\omega$ ,  $\bar{\omega}$ , respectively.  $f_{0,0}^{(0)}(z)$  is that for the KK-zero-mode.

### 3 Background field method and mass-squared eigenvalues

Our purpose is to calculate the 1-loop correction to the two- and four-point functions with vanishing external momenta with respect to the Higgs and  $W_\mu^\pm$  fields, namely the quantum correction to the coefficients  $\mu^2$ ,  $\lambda$  and  $\kappa$ , denoted by  $\delta\mu^2$ ,  $\delta\lambda$  and  $\delta\kappa$ , respectively. For simplicity, in this paper we focus on the quantum correction due to the scalar matter field  $\Phi$ .

For that purpose we use background field method, treating not only the higgs field but also  $W_\mu^\pm$  as constant fields. We then calculate bubble diagram of the scalar field under the influence of the background fields, in order to get the effective potential concerning the background fields. Finally, we can read off the quantum corrections  $\delta\mu^2$ ,  $\delta\lambda$  and  $\delta\kappa$ , by reading off the coefficients of the relevant operators in the Taylor-expansion of the effective potential.

The background 4D gauge field of our interest is written as

$$A_\mu^{cl} = \begin{pmatrix} 0 & \frac{W_\mu^+}{\sqrt{2}} & 0 \\ \frac{W_\mu^-}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (3.1)$$

while the background 4D scalar field is written as

$$\begin{aligned} A_z^{cl} &= \frac{a}{\sqrt{2}gR} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \\ A_{\bar{z}}^{cl} &= (A_z^{cl})^\dagger = \frac{a}{\sqrt{2}gR} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \end{aligned} \quad (3.2)$$

where  $A_z \equiv \frac{1}{2}(A_5 - iA_6)$ ,  $A_{\bar{z}} = A_z^\dagger$  and  $a$  is a dimensionless real field defined by use of the  $h^0$  as follows:

$$a = \frac{1}{\sqrt{2}} g |h^0| R, \quad (3.3)$$

Note that by a suitable re-phasing  $h^0$  can be represented by  $|h^0|$ .

Under the presence of the background fields the bi-linear term of the scalar  $\Phi$  is written as

$$(D_M^{cl}\Phi)^\dagger (D^{cl,M}\Phi) = -\Phi^\dagger D_M^{cl} D^{cl,M}\Phi = -\Phi^\dagger (D_\mu^{cl} D^{cl,\mu} - D_z^{cl} D^{cl,z} - D_{\bar{z}}^{cl} D^{cl,\bar{z}})\Phi$$

$$= -\Phi^\dagger(D_\mu^{cl}D^{cl,\mu} - 2D_z^{cl}D_{\bar{z}}^{cl} - 2D_{\bar{z}}^{cl}D_z^{cl})\Phi, \quad (3.4)$$

where

$$D_\mu^{cl} = \partial_\mu + igA_\mu^{cl}, \quad D_z^{cl} = \partial_z + igA_z^{cl}, \quad D_{\bar{z}}^{cl} = \partial_{\bar{z}} + igA_{\bar{z}}^{cl}, \quad (3.5)$$

with  $\partial_z \equiv \frac{1}{2}(\partial_5 - i\partial_6)$ , etc.

Substituting (2.5), (2.7) in (3.4) and by performing integral over  $x^\mu$  and  $x_5, x_6$  together with the orthonormal conditions for the mode-functions (2.7), we realize that the successive operations of the covariant derivatives to the 4D fields with definite  $(p^\mu, m, n)$

$$\begin{pmatrix} \phi_{n,m}^{(1)}(p) \\ \phi_{n,m}^{(2)}(p) \\ \phi_{n,m}^{(3)}(p) \end{pmatrix} \quad (3.6)$$

is equivalent to the multiplications of the following matrices:

$$\begin{aligned} D_\mu^{cl}D^{cl,\mu} &= - \begin{pmatrix} (p^\mu p_\mu + \frac{g^2}{2}W^{+\mu}W_\mu^-) & \sqrt{2}gp^\mu W_\mu^+ & 0 \\ \sqrt{2}gp^\mu W_\mu^- & (p^\mu p_\mu + \frac{g^2}{2}W^{+\mu}W_\mu^-) & 0 \\ 0 & 0 & p^\mu p_\mu \end{pmatrix}, \\ D_z^{cl}D_{\bar{z}}^{cl} &= - \begin{pmatrix} \frac{1}{3R^2}(n^2 + nm + m^2) & 0 & 0 \\ 0 & \frac{1}{3R^2}(n^2 + nm + m^2) & \frac{\sqrt{2}a}{4R^2}(n - \frac{n+2m}{\sqrt{3}}i) \\ 0 & \frac{\sqrt{2}a}{4R^2}(n + \frac{n+2m}{\sqrt{3}}i) & \frac{1}{3R^2}(n^2 + nm + m^2) + \frac{a^2}{2R^2} \end{pmatrix}, \\ D_{\bar{z}}^{cl}D_z^{cl} &= - \begin{pmatrix} \frac{1}{3R^2}(n^2 + nm + m^2) & 0 & 0 \\ 0 & \frac{1}{3R^2}(n^2 + nm + m^2) + \frac{a^2}{2R^2} & \frac{\sqrt{2}a}{4R^2}(n - \frac{n+2m}{\sqrt{3}}i) \\ 0 & \frac{\sqrt{2}a}{4R^2}(n + \frac{n+2m}{\sqrt{3}}i) & \frac{1}{3R^2}(n^2 + nm + m^2) \end{pmatrix}. \end{aligned} \quad (3.7)$$

Namely,

$$-(D_\mu^{cl}D^{cl,\mu} - 2D_z^{cl}D_{\bar{z}}^{cl} - 2D_{\bar{z}}^{cl}D_z^{cl}) = (p^\mu p_\mu - M^2)I_3 - \mathcal{M}^2, \quad (3.8)$$

where  $I_3$  is the  $3 \times 3$  unit matrix and

$$\mathcal{M}^2 \equiv \begin{pmatrix} -\frac{g^2}{2}W^{+\mu}W_\mu^- + M_{n,m}^2 & -\sqrt{2}gp^\mu W_\mu^+ & 0 \\ -\sqrt{2}gp^\mu W_\mu^- & -\frac{g^2}{2}W^{+\mu}W_\mu^- + M_{n,m}^2 + \frac{a^2}{R^2} & \frac{\sqrt{2}a}{R^2}(n - \frac{n+2m}{\sqrt{3}}i) \\ 0 & \frac{\sqrt{2}a}{R^2}(n + \frac{n+2m}{\sqrt{3}}i) & M_{n,m}^2 + \frac{a^2}{R^2} \end{pmatrix} \quad (3.9)$$

with

$$M_{n,m}^2 \equiv \frac{4}{3R^2}(n^2 + nm + m^2). \quad (3.10)$$

By a suitable re-phasing of  $\phi_{n,m}^{(3)}(p)$ , the matrix  $\mathcal{M}^2$  is brought to

$$\mathcal{M}^2 \equiv \begin{pmatrix} -\frac{g^2}{2}W^{+\mu}W_\mu^- + M_{n,m}^2 & -\sqrt{2}gp^\mu W_\mu^+ & 0 \\ -\sqrt{2}gp^\mu W_\mu^- & -\frac{g^2}{2}W^{+\mu}W_\mu^- + M_{n,m}^2 + \frac{a^2}{R^2} & \frac{\sqrt{2}a}{R}M_{n,m} \\ 0 & \frac{\sqrt{2}a}{R}M_{n,m} & M_{n,m}^2 + \frac{a^2}{R^2} \end{pmatrix}. \quad (3.11)$$

In order to use the background field method to get the effective potential, we need three eigenvalues of the matrix  $\mathcal{M}^2$ . Since what we are interested in are the quantum corrections to the operators in (1.6), we retain only the terms up to quadratic in  $W_\mu^\pm$ . So  $W^{+\mu}W_\mu^-$  and  $p^\mu W_\mu^+$  may be treated



as if they were small perturbations and we can rather easily get approximated eigenvalues up to the order, by using perturbative method.

One way to do is to write each eigenvalue as  $\lambda_i = \lambda_i^{(0)} + \epsilon_i$  ( $i = 1, 2, 3$ ), where  $\lambda_i^{(0)}$  is the eigenvalues for the vanishing  $W_\mu^\pm$  and the  $\epsilon_i$  is the small perturbation of each eigenvalue, and solve for  $\epsilon_i$  keeping only the terms up to  $\mathcal{O}(\epsilon)$  in the equation. Or, we may use the well-known wisdom in the quantum mechanics to get the energy eigenvalues by use of perturbative method, such as  $\langle n|H'|n\rangle$ ,  $\sum_{m \neq n} \frac{|\langle n|H'|m\rangle|^2}{E_n^{(0)} - E_m^{(0)}}$  for the first and second orders of perturbation of energy eigenvalues.

We have used two methods and have confirmed that the two methods give the same result. We will skip the detail of the derivation of the eigenvalues of  $\mathcal{M}^2$  and just give the results below. First, three eigenvalues without perturbation,  $\lambda_i^{(0)}$ , are

$$\begin{aligned}\lambda_1^{(0)} &= M_{n,m}^2, \\ \lambda_2^{(0)} &= M_{n,m}^2 + \frac{a^2}{R^2} + \frac{\sqrt{2}a}{R}M_{n,m}, \\ \lambda_3^{(0)} &= M_{n,m}^2 + \frac{a^2}{R^2} - \frac{\sqrt{2}a}{R}M_{n,m}.\end{aligned}\tag{3.12}$$

Then the eigenvalues up to the  $\mathcal{O}(W^+W^-)$  are given as

$$\begin{aligned}\lambda_1 &= M_{n,m}^2 + \frac{g^2|p^\mu W_\mu^+|^2}{M_{n,m}^2 - \frac{a^2}{2R^2}} - \frac{g^2}{2}W^{+\mu}W_\mu^- \\ &= \lambda_1^{(0)} + \left( \frac{1}{\lambda_1^{(0)} - \lambda_2^{(0)}} + \frac{1}{\lambda_1^{(0)} - \lambda_3^{(0)}} \right) g^2|p^\mu W_\mu^+|^2 - \frac{g^2}{2}W^{+\mu}W_\mu^-, \\ \lambda_2 &= M_{n,m}^2 + \frac{a^2}{R^2} + \frac{\sqrt{2}a}{R}M_{n,m} + \frac{g^2|p^\mu W_\mu^+|^2}{\frac{a^2}{R^2} + \frac{\sqrt{2}a}{R}M_{n,m}} - \frac{g^2}{4}W^{+\mu}W_\mu^- \\ &= \lambda_2^{(0)} + \left( \frac{1}{\lambda_2^{(0)} - \lambda_1^{(0)}} \right) g^2|p^\mu W_\mu^+|^2 - \frac{g^2}{4}W^{+\mu}W_\mu^-, \\ \lambda_3 &= M_{n,m}^2 + \frac{a^2}{R^2} - \frac{\sqrt{2}a}{R}M_{n,m} + \frac{g^2|p^\mu W_\mu^+|^2}{\frac{a^2}{R^2} - \frac{\sqrt{2}a}{R}M_{n,m}} - \frac{g^2}{4}W^{+\mu}W_\mu^- \\ &= \lambda_3^{(0)} + \left( \frac{1}{\lambda_3^{(0)} - \lambda_1^{(0)}} \right) g^2|p^\mu W_\mu^+|^2 - \frac{g^2}{4}W^{+\mu}W_\mu^-.\end{aligned}\tag{3.13}$$

## 4 Quantum corrections

We now obtain the quantum corrections  $\delta\mu^2$ ,  $\delta\lambda$  and  $\delta\kappa$ , by calculating the effective potential as a function of the background fields and reading off the suitable coefficients in the Taylor expansion of the effective potential with respect to the background fields, or equivalently  $a$  and  $W^\pm$ .

The effective potential is given by the following formula:

$$V_{eff}(a, W) = \int \frac{d^4 p_E}{(2\pi)^4} \sum_{n,m} [\ln(p_E^2 + M^2 + \lambda_1) + \ln(p_E^2 + M^2 + \lambda_2) + \ln(p_E^2 + M^2 + \lambda_3)], \tag{4.1}$$



where  $p_E$  is a Euclidean momentum and accordingly the gauge field  $W^\pm$  should be Wick-rotated and the replacement

$$W^{+\mu}W_\mu^- \rightarrow -W^+ \cdot W^-, \quad p_\mu W^{+\mu} \rightarrow -p_E \cdot W^+ \quad (4.2)$$

is understood. For instance

$$\begin{aligned} \lambda_1 &\rightarrow M_{n,m}^2 + \frac{g^2 |p_E \cdot W^+|^2}{M_{n,m}^2 - \frac{a^2}{2R^2}} + \frac{g^2}{2} W^+ \cdot W^- \\ &= \lambda_1^{(0)} + \left( \frac{1}{\lambda_1^{(0)} - \lambda_2^{(0)}} + \frac{1}{\lambda_1^{(0)} - \lambda_3^{(0)}} \right) g^2 |p_E \cdot W^+|^2 + \frac{g^2}{2} W^+ \cdot W^-. \end{aligned} \quad (4.3)$$

What we are interested in are the operators

$$a^2, \quad a^4, \quad a^2 W^{+\mu} W_\mu^-. \quad (4.4)$$

We will discuss the quantum corrections to these operators successively below.

## 4.1 The $a^2$ term

First we calculate the  $a^2$  term of the effective potential. This operator is expected to be not induced even at the quantum level at least as a local operator (except for the contribution due to the Wilson loop), and therefore is expected to be UV-finite.

We set  $W^\pm = 0$ , as we are interested in the operator  $a^2$  that does not contain the field  $W$ . Then, we find that only the terms with  $\lambda_{2,3}$  in (4.1), depending on  $a$ , contribute to this operator. Though each of  $\lambda_{2,3}$  has a term linear in  $M_{n,m}$ , the combined contributions can be written in terms of  $M_{n,m}^2$ :

$$\begin{aligned} &\ln(p_E^2 + M^2 + \lambda_2) + \ln(p_E^2 + M^2 + \lambda_3) \\ &= \ln \left\{ (p_E^2 + M^2 + M_{n,m}^2)^2 + 2(p_E^2 + M^2) \frac{a^2}{R^2} + \frac{a^4}{R^4} \right\}. \end{aligned} \quad (4.5)$$

Then the  $a^2$  term in the Taylor-expansion is easily found to be

$$2 \frac{p_E^2 + M^2}{(p_E^2 + M^2 + M_{n,m}^2)^2} \frac{a^2}{R^2}. \quad (4.6)$$

Thus the induced  $a^2$  operator at the quantum level can be written as

$$2 \frac{a^2}{R^2} \int \frac{d^4 p_E}{(2\pi)^4} \sum_{n,m} \frac{p_E^2 + M^2}{(p_E^2 + M^2 + M_{n,m}^2)^2}. \quad (4.7)$$

By using formulae,

$$\frac{1}{\alpha^2} = \int_0^\infty t e^{-\alpha t} dt, \quad (4.8)$$

$$\sum_{n,m} e^{-t M_{n,m}^2} = \frac{\sqrt{3}\pi R^2}{2} \sum_{k,l} \frac{1}{t} e^{-\frac{(\pi R)^2 (k^2 + kl + l^2)}{t}} \quad (\text{Poisson resummation}), \quad (4.9)$$

(4.7) can be written in a form,

$$\sqrt{3}\pi a^2 \int_0^\infty dt \int \frac{d^4 p_E}{(2\pi)^4} (p_E^2 + M^2) e^{-t(p_E^2 + M^2)} \sum_{k,l} e^{-\frac{(\pi R)^2 (k^2 + kl + l^2)}{t}}. \quad (4.10)$$

In order to see whether UV-divergence is absent, we focus on the “zero-winding” sector, i.e.  $k = l = 0$ . Then the integral over  $t$  is easily done and the remaining integral is

$$\int_0^\infty dt \int \frac{d^4 p_E}{(2\pi)^4} (p_E^2 + M^2) e^{-t(p_E^2 + M^2)} = \int \frac{d^4 p_E}{(2\pi)^4} \times 1. \quad (4.11)$$

Though (4.11) is superficially UV-divergent, we have to be a little careful about the treatment, since a momentum cutoff violates gauge symmetry. So we invoke dimensional regularization method, by changing  $d^4 p_E$  to  $d^d p_E$  ( $d$ : space-time dimension) and taking  $d \rightarrow 4$  at the final stage. As the matter of fact, we find that (4.11) just vanishes, as we expected. Namely,

$$\int \frac{d^d p_E}{(2\pi)^d} \times 1 = \int \frac{d^d p_E}{(2\pi)^d} (p_E^2 + M^2)^0 = \frac{1}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(-\frac{d}{2})}{\Gamma(0)} M^d = 0, \quad (4.12)$$

since  $\Gamma(0) = \infty$ .

## 4.2 The $a^4$ term

We now calculate the  $a^4$  term, in a similar way as what we took in the calculation of the quadratic term  $a^2$ . Again we focus on (4.5) to get the  $a^4$  term:

$$\frac{a^4}{R^4} \int \frac{d^4 p_E}{(2\pi)^4} \sum_{n,m} \left\{ \frac{1}{(p_E^2 + M^2 + M_{n,m}^2)^2} - 2 \frac{(p_E^2 + M^2)^2}{(p_E^2 + M^2 + M_{n,m}^2)^4} \right\}. \quad (4.13)$$

By using (4.8), (4.9), together with

$$\frac{1}{\alpha^4} = \frac{1}{6} \int_0^\infty t^3 e^{-\alpha t} dt, \quad (4.14)$$

(4.13) can be put into a form

$$\frac{\sqrt{3}}{2} \pi \frac{a^4}{R^2} \int_0^\infty dt \int \frac{d^4 p_E}{(2\pi)^4} e^{-t(p_E^2 + M^2)} \left\{ 1 - \frac{1}{3} (p_E^2 + M^2)^2 t^2 \right\} \sum_{k,l} e^{-\frac{(\pi R)^2 (k^2 + kl + l^2)}{t}}. \quad (4.15)$$

To see the UV-divergence, we focus on the zero-winding sector. Then the integral over  $t$  is easily done by use of formulae

$$\int_0^\infty dt e^{-t(p_E^2 + M^2)} = \frac{1}{p_E^2 + M^2}, \quad (4.16)$$

$$\int_0^\infty dt e^{-t(p_E^2 + M^2)} t^2 = \frac{2}{(p_E^2 + M^2)^3}. \quad (4.17)$$

Namely,

$$\int_0^\infty dt e^{-t(p_E^2 + M^2)} \left\{ 1 - \frac{1}{3} (p_E^2 + M^2)^2 t^2 \right\} = \frac{1}{3} \frac{1}{p_E^2 + M^2}. \quad (4.18)$$

Thus the zero-winding sector of (4.15) can be written as

$$\frac{\sqrt{3}}{6} \pi \frac{a^4}{R^2} \int \frac{d^4 p_E}{(2\pi)^4} \frac{1}{p_E^2 + M^2}. \quad (4.19)$$

This time (4.19) is apparently UV-divergent even if we utilize the dimensional regularization method:

$$\frac{\sqrt{3}}{6} \pi \frac{a^4}{R^2} \frac{\Gamma(1 - \frac{d}{2})}{(4\pi)^{\frac{d}{2}}} M^{d-2} \quad (d \rightarrow 4). \quad (4.20)$$

### 4.3 The $a^2 W^{+\mu} W_\mu^-$ term

The  $a^2 W^{+\mu} W_\mu^-$  term originates from  $W^+ \cdot W^-$  and  $|p_E \cdot W^+|^2$  terms in the eigenvalues  $\lambda_{1,2,3}$ .

We first discuss the term linear in  $W^+ \cdot W^-$ . Extracting only the linear term,

$$\begin{aligned} & \ln(p_E^2 + M^2 + \lambda_2) + \ln(p_E^2 + M^2 + \lambda_3) \\ & \rightarrow \frac{g^2}{4} W^+ \cdot W^- \left\{ \frac{1}{p_E^2 + M^2 + \lambda_2^{(0)}} + \frac{1}{p_E^2 + M^2 + \lambda_3^{(0)}} \right\} \end{aligned} \quad (4.21)$$

Let us note that the second line of the above equation just corresponds to the 1-loop Feynman diagram due to the 4-point vertex with respect to the fields  $W^+$ ,  $W^-$  and the scalar matter fields  $\phi_{n,m}^{(2,3)}(p)$  (with one propagator for the scalar fields).

Now in (4.21), we retain only the quadratic term in  $a$ :

$$\begin{aligned} & \ln(p_E^2 + M^2 + \lambda_2) + \ln(p_E^2 + M^2 + \lambda_3) \\ & \rightarrow \frac{g^2}{2} \frac{a^2}{R^2} W^+ \cdot W^- \left\{ \frac{1}{(p_E^2 + M^2 + M_{n,m}^2)^2} - 2 \frac{p_E^2 + M^2}{(p_E^2 + M^2 + M_{n,m}^2)^3} \right\}. \end{aligned} \quad (4.22)$$

Secondly, we discuss the term linear in  $|p_E \cdot W^+|^2$ . Extracting only the linear term,

$$\begin{aligned} & \ln(p_E^2 + M^2 + \lambda_1) + \ln(p_E^2 + M^2 + \lambda_2) + \ln(p_E^2 + M^2 + \lambda_3) \\ & \rightarrow g^2 |p_E \cdot W^+|^2 \left\{ \left( \frac{1}{\lambda_1^{(0)} - \lambda_2^{(0)}} + \frac{1}{\lambda_1^{(0)} - \lambda_3^{(0)}} \right) \frac{1}{p_E^2 + M^2 + \lambda_1^{(0)}} \right. \\ & \quad \left. + \frac{1}{\lambda_2^{(0)} - \lambda_1^{(0)}} \frac{1}{p_E^2 + M^2 + \lambda_2^{(0)}} + \frac{1}{\lambda_3^{(0)} - \lambda_1^{(0)}} \frac{1}{p_E^2 + M^2 + \lambda_3^{(0)}} \right\} \\ & = -g^2 |p_E \cdot W^+|^2 \frac{1}{p_E^2 + M^2 + \lambda_1^{(0)}} \left( \frac{1}{p_E^2 + M^2 + \lambda_2^{(0)}} + \frac{1}{p_E^2 + M^2 + \lambda_3^{(0)}} \right). \end{aligned} \quad (4.23)$$

The last line of (4.23) just corresponds to the 1-loop Feynman diagrams due to the 3-point vertex with respect to  $W^\pm$  and two scalar fields  $\phi_{n,m}^{(1)}(p)$  and  $\phi_{n,m}^{(2,3)}(p)$  (with two propagators of these scalars).

Under  $p_E$  integration, done later on, the following replacement can be justified:

$$|p_E \cdot W^+|^2 \rightarrow \frac{p_E^2}{d} W^+ \cdot W^-, \quad (4.24)$$

assuming dimensional regularization. Then, (4.23) reduces to

$$-g^2 W^+ \cdot W^- \frac{p_E^2}{d} \frac{1}{p_E^2 + M^2 + \lambda_1^{(0)}} \left( \frac{1}{p_E^2 + M^2 + \lambda_2^{(0)}} + \frac{1}{p_E^2 + M^2 + \lambda_3^{(0)}} \right). \quad (4.25)$$

Again, retaining only the term quadratic in  $a$ , we get

$$-2g^2 \frac{a^2}{R^2} W^+ \cdot W^- \frac{p_E^2}{d} \left\{ \frac{1}{(p_E^2 + M^2 + M_{n,m}^2)^3} - 2 \frac{p_E^2 + M^2}{(p_E^2 + M^2 + M_{n,m}^2)^4} \right\}. \quad (4.26)$$

Putting (4.22) and (4.26) together we get

$$g^2 \frac{a^2}{R^2} W^{+\mu} W_\mu^- \left\{ -\frac{1}{2} \frac{1}{(p_E^2 + M^2 + M_{n,m}^2)^2} + \frac{(1 + \frac{2}{d})p_E^2 + M^2}{(p_E^2 + M^2 + M_{n,m}^2)^3} - \frac{4}{d} \frac{p_E^2(p_E^2 + M^2)}{(p_E^2 + M^2 + M_{n,m}^2)^4} \right\}, \quad (4.27)$$

where a replacement  $W^+ \cdot W^- \rightarrow -W^{+\mu} W_\mu^-$  has been done.

Thus, the quantum correction to the  $\frac{a^2}{R^2} W^{+\mu} W_\mu^-$  operator can be written as

$$g^2 \frac{a^2}{R^2} W^{+\mu} W_\mu^- \int \frac{d^d p_E}{(2\pi)^d} \sum_{n,m} \left\{ -\frac{1}{2} \frac{1}{(p_E^2 + M^2 + M_{n,m}^2)^2} + \frac{(1 + \frac{2}{d})p_E^2 + M^2}{(p_E^2 + M^2 + M_{n,m}^2)^3} - \frac{4}{d} \frac{p_E^2(p_E^2 + M^2)}{(p_E^2 + M^2 + M_{n,m}^2)^4} \right\}. \quad (4.28)$$

Using another formula

$$\frac{1}{\alpha^3} = \frac{1}{2} \int_0^\infty t^2 e^{-t\alpha} dt, \quad (4.29)$$

(4.28) can be put in a form after performing Poisson resummation,

$$\begin{aligned} & \frac{\sqrt{3}\pi}{2} g^2 a^2 W^{+\mu} W_\mu^- \int_0^\infty dt \int \frac{d^d p_E}{(2\pi)^d} e^{-t(p_E^2 + M^2)} \left\{ -\frac{1}{2} + \frac{t}{2} \left[ \left(1 + \frac{2}{d}\right) p_E^2 + M^2 \right] - \frac{2t^2}{3d} p_E^2 (p_E^2 + M^2) \right\} \\ & \times \sum_{k,l} e^{-\frac{(\pi R)^2 (k^2 + kl + l^2)}{t}}. \end{aligned} \quad (4.30)$$

By use of a formula

$$\int_0^\infty dt e^{-t(p_E^2 + M^2)} t = \frac{1}{(p_E^2 + M^2)^2}, \quad (4.31)$$

together with (4.16), (4.17), the zero-winding sector of (4.30) turns out to take a form

$$\frac{\sqrt{3}\pi}{2} g^2 a^2 W^{+\mu} W_\mu^- \int \frac{d^d p_E}{(2\pi)^d} \left( -\frac{1}{3d} \right) \frac{p_E^2}{(p_E^2 + M^2)^2}. \quad (4.32)$$

By utilizing dimensional regularization method the zero-winding sector is written as

$$-\frac{\sqrt{3}}{12} \pi g^2 a^2 W^{+\mu} W_\mu^- \frac{\Gamma(1 - \frac{d}{2})}{(4\pi)^{\frac{d}{2}}} M^{d-2} \quad (d \rightarrow 4). \quad (4.33)$$

## 4.4 Divergent parts of the quantum corrections

We have seen that at the classical level

$$\lambda_{tree} = \kappa_{tree} = \frac{1}{2} g^2. \quad (4.34)$$

Note that the relation  $M_H = 2M_W$  at the classical level is the consequence of the relation  $\lambda_{tree} = \kappa_{tree}$ . Now we will see whether the UV-divergent parts of  $\lambda$  and  $\kappa$  still preserve this relation, so that the deviation from the relation  $M_H = 2M_W$  can be calculated as a finite value.

The divergent parts of  $\lambda$  and  $\kappa$ , defined by  $\delta\lambda^{div}$ ,  $\delta\kappa^{div}$  can be easily read off by replacing  $a$  by  $h_0$  in (4.20) and (4.33), according to the relation (3.3), and changing the overall sign (the effective potential contributes to the effective lagrangian with opposite sign). Namely, we find

$$\begin{aligned}\delta\lambda^{div} &= \frac{\sqrt{3}}{24}\pi g^4 R^2 M^{d-2} \frac{\Gamma(1 - \frac{d}{2})}{(4\pi)^{\frac{d}{2}}}, \\ \delta\kappa^{div} &= \frac{\sqrt{3}}{24}\pi g^4 R^2 M^{d-2} \frac{\Gamma(1 - \frac{d}{2})}{(4\pi)^{\frac{d}{2}}}.\end{aligned}\tag{4.35}$$

We thus find  $\delta\lambda^{div} = \delta\kappa^{div}$  as we expected. Let us note that quantum correction  $\delta\mu^2$  is UV-finite by itself as we have seen in (4.12).

## 5 Calculable two observables

The recent LHC experiments [1], [2] have now determined the Higgs mass as  $M_H = 126$  GeV:

$$M_H^2 = 126^2 \text{ GeV}^2 = 1.59 \times 10^4 \text{ GeV}^2,\tag{5.1}$$

$$\left(\frac{M_H}{2M_W}\right)^2 = \left(\frac{126}{160}\right)^2 = 0.620 \rightarrow \Delta \equiv \left(\frac{M_H}{2M_W}\right)^2 - 1 = -0.380.\tag{5.2}$$

A remarkable thing in our model is that both of these observables  $M_H^2$ ,  $\Delta$  are calculable (as finite values without need of renormalization procedure) in terms of fundamental parameters of the theory,  $R$  and  $M$ . In fact,

$$M_H^2 = 2\delta\mu^2,\tag{5.3}$$

$$\Delta = \frac{\lambda}{\kappa} - 1 = \frac{\frac{g^2}{2} + \delta\lambda}{\frac{g^2}{2} + \delta\kappa} - 1 \simeq \frac{2}{g^2}(\delta\lambda - \delta\kappa)\tag{5.4}$$

are both finite at least at the 1-loop level, thanks to the key relation  $\delta\lambda^{div} = \delta\kappa^{div}$  (see 4.35). We now derive the finite expressions for  $M_H^2$  and  $\Delta$ .

The quantum corrections  $\delta\mu^2$ ,  $\delta\lambda$ ,  $\delta\kappa$  are obtained from (4.10), (4.15) and (4.30) by utilizing (3.3). Namely,

$$\delta\mu^2 = -\frac{\sqrt{3}}{2}\pi g^2 R^2 \int_0^\infty dt \int \frac{d^4 p_E}{(2\pi)^4} (p_E^2 + M^2) e^{-t(p_E^2 + M^2)} \sum_{(k,l) \neq (0,0)} e^{-\frac{(\pi R)^2(k^2 + kl + l^2)}{t}}\tag{5.5}$$

$$\delta\lambda = \frac{\sqrt{3}}{8}\pi g^4 R^2 \int_0^\infty dt \int \frac{d^d p_E}{(2\pi)^d} e^{-t(p_E^2 + M^2)} \left\{ 1 - \frac{1}{3}(p_E^2 + M^2)^2 t^2 \right\} \sum_{k,l} e^{-\frac{(\pi R)^2(k^2 + kl + l^2)}{t}}\tag{5.6}$$

$$\begin{aligned}\delta\kappa &= -\frac{\sqrt{3}}{4}\pi g^4 R^2 \int_0^\infty dt \int \frac{d^d p_E}{(2\pi)^d} e^{-t(p_E^2 + M^2)} \left\{ -\frac{1}{2} + \frac{t}{2} \left[ \left(1 + \frac{2}{d}\right) p_E^2 + M^2 \right] - \frac{2t^2}{3d} p_E^2 (p_E^2 + M^2) \right\} \\ &\times \sum_{k,l} e^{-\frac{(\pi R)^2(k^2 + kl + l^2)}{t}},\end{aligned}\tag{5.7}$$

where  $d = 4$  is understood for  $\delta\mu^2$ , since we know that this is UV-finite, while  $d$  has been left arbitrary for  $\delta\lambda$  and  $\delta\kappa$ , since they are UV-divergent. The difference  $\delta\lambda - \delta\kappa$  is UV-finite and is given by setting

$d = 4$  as

$$\delta\lambda - \delta\kappa = \frac{\sqrt{3}}{8}\pi g^4 R^2 \int_0^\infty dt \int \frac{d^4 p_E}{(2\pi)^4} e^{-t(p_E^2 + M^2)} \times \left\{ -\frac{2}{3}t^2(p_E^2 + M^2)^2 + \left(\frac{1}{3}M^2 t^2 + \frac{3}{2}t\right)(p_E^2 + M^2) - \frac{1}{2}M^2 t \right\} \sum_{(k,l) \neq (0,0)} e^{-\frac{(\pi R)^2(k^2 + kl + l^2)}{t}}. \quad (5.8)$$

By performing the integration over  $p_E$  and by changing the integration variable as  $\frac{R^2}{t} = u$ , the finite expressions of (5.3) and (5.4) are given as

$$M_H^2 = -\frac{\sqrt{3}}{16\pi}g^2 \frac{1}{R^2} \sum_{(k,l) \neq (0,0)} \int_0^\infty du (2u + \hat{M}^2) e^{-\frac{\hat{M}^2}{u}} e^{-\pi^2(k^2 + kl + l^2)u}, \quad (5.9)$$

$$\Delta = -\frac{\sqrt{3}}{64\pi}g^2 \sum_{(k,l) \neq (0,0)} \int_0^\infty du \left(1 + \frac{\hat{M}^2}{u} + \frac{1}{3}\frac{\hat{M}^4}{u^2}\right) e^{-\frac{\hat{M}^2}{u}} e^{-\pi^2(k^2 + kl + l^2)u}, \quad (5.10)$$

where  $\hat{M} \equiv RM$  is a dimensionless parameter.

For a specific case of  $\hat{M} = 0$ , the integral over  $u$  can be easily performed directly or by use of the definition of gamma functions, and (5.9) and (5.10) reduce to simple expressions

$$M_H^2 = -\frac{\sqrt{3}}{8\pi^5}g^2 \frac{1}{R^2} \sum_{(k,l) \neq (0,0)} \frac{1}{(k^2 + kl + l^2)^2}, \quad (5.11)$$

$$\Delta = -\frac{\sqrt{3}}{64\pi^3}g^2 \sum_{(k,l) \neq (0,0)} \frac{1}{k^2 + kl + l^2}. \quad (5.12)$$

Let us note (5.11) is finite while the sum over  $k, l$  in (5.12) is divergent. In fact, roughly speaking in the sum  $\sum_{(k,l) \neq (0,0)} \frac{1}{k^2 + kl + l^2}$  the contribution from the region of large  $k, l$  behaves as an integral

$$\sum_{(k,l) \neq (0,0)} \frac{1}{k^2 + kl + l^2} \sim \int \frac{dx_5 dx_6}{x_5^2 + x_5 x_6 + x_6^2}, \quad (5.13)$$

which is logarithmically divergent (as the contribution from the region of large  $x_{5,6}$ ). This logarithmic divergence comes from the region of larger  $k, l$  and therefore is a sort of IR-divergence. In fact, we easily see that (4.13) has an IR divergence coming from the contribution of the zero-KK-mode sector  $n = m = 0$  for the case of  $M = 0$ , while (4.28) does not have for  $d = 4$ . Thus  $\delta\lambda - \delta\kappa$  should have an IR divergence.

We thus find that non-vanishing  $M$  is necessary to avoid the IR divergence. This argument, in turn, suggests that when  $\hat{M}$  is small,  $\Delta$  behaves as  $\propto \log \hat{M}$ , in order to be consistent with the logarithmic IR-divergence.

## 6 Numerical analysis of $M_H^2$ and $\Delta$

Though  $M_H^2$  and  $\Delta$  given in (5.9) and (5.10) are calculable as finite values, they cannot be obtained analytically. Thus, in this section we perform some numerical analysis. The purpose here is to see

whether this toy model is roughly able to realize the (absolute values) of observed values (5.1) and (5.2) for  $M_H^2$  and  $\Delta$  for suitable choices of  $R$  and  $\hat{M}$ , even if the signs of these two quantities cannot be correctly reproduced. Let us note that (5.9) and (5.10) are of the same sign, while (5.1) and (5.2) tell us they have opposite signs.

From such a point of view, it may be useful to note that  $|\Delta| = 0.380$  in (5.2) is greater than what we naively expect as a 1-loop quantum correction, i.e. a value roughly of the order  $\alpha$ , while  $M_H^2$  in (5.1) can be naturally realized by a choice of the compactification scale  $M_c = 1/R$  of the order of 1-10 TeV. Fortunately, we have a mechanism to realize such “sizable”  $|\Delta|$ . Namely, because of the IR-singularity, we expect that for sufficiently small  $\hat{M}$

$$|\Delta| \propto -\ln \hat{M}. \quad (6.1)$$

Thus choosing suitably small  $\hat{M}$  observed  $\Delta$  should be realized. (It is interesting to note that at least in 5D theory with orbifold compactification, small “ $Z_2$ -odd” bulk masses correspond to large fermion masses of the order of weak scale, such as the top quark mass.)

From now on we thus assume that  $\hat{M}$  is small enough and will confirm the expectation mentioned above. We first discuss  $|\Delta|$ .

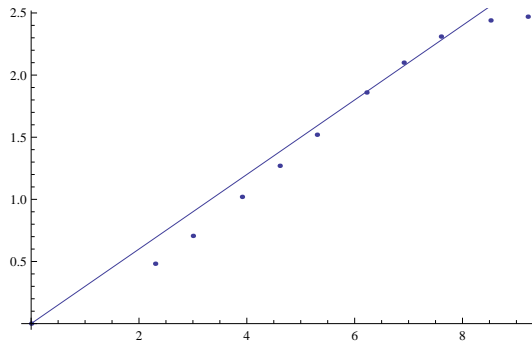
We have performed a numerical computation of the following factor in (5.10):

$$f(\hat{M}^2) \equiv \sum_{(k,l) \neq (0,0)} \int_0^\infty du \left( 1 + \frac{\hat{M}^2}{u} + \frac{1}{3} \frac{\hat{M}^4}{u^2} \right) e^{-\frac{\hat{M}^2}{u}} e^{-\pi^2(k^2+kl+l^2)u}. \quad (6.2)$$

Actually, because of the lack of the computational capability we have in our hand, we have approximated  $f(\hat{M})$  by the following function

$$\bar{f}(\hat{M}^2) \equiv \sum_{(k,l) \neq (0,0), |k|, |l| \leq 30} \int_{0.00015}^{100} du \left( 1 + \frac{\hat{M}^2}{u} + \frac{1}{3} \frac{\hat{M}^4}{u^2} \right) e^{-\frac{\hat{M}^2}{u}} e^{-\pi^2(k^2+kl+l^2)u}. \quad (6.3)$$

The result of the numerical calculation for the function  $\bar{f}(\hat{M})$  is shown in Fig.1.



**Figure 1:** The function  $\bar{f}(\hat{M}^2)$ . The horizontal axis is  $-\ln \hat{M}^2$  and the vertical axis is  $\bar{f}(\hat{M}^2)$ . The straight line is for  $\bar{f}(\hat{M}^2) = 0.3(-\ln \hat{M}^2)$

As we expected, as  $\hat{M}^2$  becomes small enough or equivalently as  $-\ln \hat{M}^2$  becomes large enough, the function shows logarithmic behavior

$$\bar{f}(\hat{M}^2) \simeq 0.3(-\ln \hat{M}^2). \quad (6.4)$$



The reason why the function  $\bar{f}(\hat{M}^2)$  finally starts to be saturated for larger  $-\ln \hat{M}^2$  is easily understood. If the sum over  $k, l$  are taken up to arbitrarily large integers as in the original function  $f(\hat{M}^2)$ , the function should become arbitrarily large for sufficiently large  $-\ln \hat{M}^2$ . Actually, however, in the approximated function  $\bar{f}(\hat{M}^2)$ , the sum over  $k, l$  are only up to  $|k| = |l| = 30$ , not infinity. Thus the function  $\bar{f}(\hat{M}^2)$  never exceeds

$$\frac{1}{\pi^2} \sum_{(k,l) \neq (0,0), |k|, |l| \leq 30} \frac{1}{k^2 + kl + l^2} = 0.341, \quad (6.5)$$

which is nothing but  $\bar{f}(0)$ , with the region of integral being replaced by  $0 \leq u < \infty$  (see (5.12)). We thus reasonably expect that the original function  $f(\hat{M}^2)$  should behave as

$$f(\hat{M}^2) \simeq 0.3(-\ln \hat{M}^2). \quad (6.6)$$

We now turn to another observable  $M_H^2$ . In this case we can use the formula (5.11), corresponding to  $\hat{M}^2 = 0$ , since the sum over  $k, l$  is finite, in contrast to the case of  $\Delta$ . A numerical computation yields

$$\sum_{(k,l) \neq (0,0)} \frac{1}{(k^2 + kl + l^2)^2} = 7.71. \quad (6.7)$$

To summarize, we have obtained

$$|M_H^2| \simeq \frac{\sqrt{3}}{8\pi^5} g^2 \frac{1}{R^2} \sum_{(k,l) \neq (0,0)} \frac{1}{(k^2 + kl + l^2)^2} = 0.0685 \frac{\alpha}{\sin^2 \theta_W} \frac{1}{R^2} = 2.2 \times 10^{-3} \frac{1}{R^2}, \quad (6.8)$$

$$|\Delta| = \frac{\sqrt{3}}{64\pi} g^2 f(\hat{M}^2) \simeq 3.25 \times 10^{-2} \frac{\alpha}{\sin^2 \theta_W} (-\ln \hat{M}^2) = 2.0 \times 10^{-3} (-\ln \hat{M}), \quad (6.9)$$

where  $\alpha = \frac{1}{137}$  and  $\sin^2 \theta_W = 0.23$  have been used. Actually, our model with SU(3) gauge group predicts  $\sin^2 \theta_W = \frac{3}{4}$ , far from 0.23. We, however, take the observed value 0.23, hoping that in a realistic model it is realized by the introduction of brane-localized kinetic term or by making the gauge group semi-simple such as SO(5)  $\times$  U(1).

By comparing these results with the observed values (5.1) and (5.2), we finally get

$$\frac{1}{R} \simeq 2.7 \text{ TeV}, \quad -\ln \hat{M} \simeq 190. \quad (6.10)$$

## 7 Summary

In this paper we addressed a question whether the recently observed Higgs mass  $M_H = 126$  GeV is calculable as a finite value in the scenario of gauge-Higgs unification (GHU). We first pointed out that the recently observed Higgs mass is of  $\mathcal{O}(M_W)$  and seems to suggest that the Higgs mass is handled by gauge interaction, roughly speaking. To be more specific, we discussed that in both scenarios of GHU (formulated on 6D space-time) and SUSY (MSSM) proposed mainly for the purpose of solving the hierarchy problem, the quartic self-coupling of the Higgs field is governed by the gauge-principle, being  $\mathcal{O}(g^2)$ . This fact led to the expectation that the deviation of the Higgs mass from the prediction

at the tree level is calculable as a finite value, free from UV-divergence, in GHU scenario, not only in 5D space-time but also in higher space-time dimensions, such as 6D. The situation is similar to the case of MSSM, where the deviation of the Higgs mass from  $(\cos \beta)M_Z$  is calculable in terms of the SUSY breaking masses, such as the stop mass and the “ $A$ -term”.

We have argued that, as a new feature of the GHU scenario, not shared by MSSM, not only the quartic self-coupling, but also the quadratic coupling of the Higgs is calculable as well, just because none of gauge invariant local operators made of higher-dimensional field strength induced at the quantum level should not have such quadratic operator.

Thus we claimed that in the GHU, as the matter of fact, we have two independent calculable observables, i.e.

$$M_H^2, \quad \Delta \equiv \left( \frac{M_H}{2M_W} \right)^2 - 1. \quad (7.1)$$

This expectation has been confirmed by explicit calculations of the quantum corrections to these quantities in a toy model. Note that in our model of GHU, both quantities just vanish at the tree level.

The model we adopted was a 6D toy model formulated on  $T^2/Z_3$  orbifold as the extra-space. For brevity, as the matter field we introduced 6D scalar fields, behaving as an  $SU(3)$  triplet, and the quantum corrections to  $M_H^2$  and  $\Delta$  due to the self-interactions of the higher-dimensional gauge fields have not been included in our analysis. Note that this treatment is consistent with gauge invariance, just because each of the bi-linear terms of scalar fields and gauge fields has gauge symmetry independently and the quantum correction due to the each sector is gauge invariant by itself.

Although the toy model is sufficient for the purpose to demonstrate that we have two calculable observables in the GHU scenario, obviously to get the realistic values for these quantities it is necessary to work in a realistic model with quarks and leptons and to incorporate the contributions due to the self-interactions of higher-dimensional gauge bosons  $A_M$ . We hope that the problem of the mutual sign in the quantum corrections to  $M_H^2$  and  $\Delta$  pointed out in this paper is solved by the calculations in such realistic framework. We would like to report on the results of the calculations in a future publication.

A comment on brane localized tadpole terms is now in order. As discussed in [9], brane localized tadpole terms like  $F_{56}$  can be allowed in a gauge invariant way if a  $U(1)$  is included in the gauge group unbroken at the orbifold fixed points. These terms yield the Higgs mass localized at the fixed points from the commutator  $g[A_5, A_6]$  and its mass is divergent in general, which spoils our argument on the calculability in this paper. However, [9] suggests that the tadpole terms can be cancelled in the case of  $T^2/Z_4$  orbifold after integrating them along the extra spatial dimensions, if we choose a suitable fermion content. Since our results are independent of the way of compactification and the choice of fermion content, the finiteness of Higgs mass still holds true and its realistic values should be obtained in the models of GHU on  $T^2/Z_4$ .

Finally, we would like to comment on the possible effect due to the wave-function renormalization in the evaluation of the quantum correction to the ratio  $\left( \frac{M_H}{2M_W} \right)^2$ . In the evaluation, we have included only the quantum corrections to the couplings  $\lambda$  and  $\kappa$ , as is seen in (5.4). What we have calculated

were essentially quantum corrections to the mass-squared of the Higgs and  $W^\pm$  gauge bosons. Let us recall that among the 3 parameters  $S, T$  and  $U$  of Peskin-Takeuchi utilized for the electro-weak precision tests, only  $T$  has a strong dependence on the masses of heavy particles, such as  $t$  quark mass, behaving roughly as  $m_t^2$ , while the  $m_t$  dependence of  $S$  or  $U$  parameters is at most  $\log m_t$ . We thus expect that the effects due to heavy particles in the wave-function renormalization are not remarkable. We also point out that, even if we calculate the effects due to the wave-function renormalization, the observable  $\Delta$  is still calculable, since the wave-function renormalizations for the Higgs and  $W^\pm$  boson are nothing but the quantum corrections to the relevant operators  $F_{\mu 5}F^{\mu 5}$ ,  $F_{\mu 6}F^{\mu 6}$  and  $F_{\mu\nu}F^{\mu\nu}$ , which are all included in a single operator  $F_{MN}F^{MN}$ , and again the divergent parts of the wave-function renormalizations for the Higgs and  $W^\pm$  are the same.

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